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1992 J. Phys. A: Math. Gen. 25 3133

(http://iopscience.iop.org/0305-4470/25/11/021)

J. Phys. A: Math. Gen. 25 (1992) 3133–3142. Printed in the UK

Exact phase transition points for vertex models on even-coordinated lattices†

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Received 10 June 1991, in final form 27 February 1992

Abstract. We study the manifold of fixed points of the generalized weak-graph transformation on lattices of even coordination number for the most general vertex model respecting spin-flip symmetry. We conjecture these fixed points to be the loci of phase transitions. As an example, we turn to coordination number six and find phase transitions in certain regions of the manifold of fixed points: first by investigating a gauge-invariant Ising model on a three-dimensional SC lattice with the help of Monte-Carlo simulations; and second for the ice-type zero-field ferroelectric model, in which the transition between frozen ordered and disordered phase is of first order.

1. Introduction

In statistical mechanics the goal is to calculate expectation values of some observables, especially in the neighbourhood of a phase transition, where the critical exponents of the model under investigation are defined. Exact results for these properties of a statistical mechanics model can be obtained if one can solve the model (e.g. diagonalize the transfer matrix) in the thermodynamic limit. Since models have to fulfil certain restrictive relations (e.g. Yang–Baxter equations) to be accessible by this program, the class of such models is very small. Computer simulations of a model for various temperatures near the expected phase transition are widely used to overcome this difficulty. However, the better the exact transition temperature (defined only in the infinite volume limit) is known, the better the desired information may be extracted from the data.

To determine the exact transition temperature without solving the model completely, one may think, roughly speaking, of the following concept: define a transformation that maps points in the parameter space of the model referring to physically similar macroscopic states onto each other. If phase transition points do not find a 'similar' point to map on, they will be fixed points under the transformation. Indeed, half a century ago, Kramers and Wannier [8] realized this program for the square-lattice Ising model and detected its transition temperature using a duality

† Work supported in part by the Sonderforschungsbereich 341, Köln-Aachen-Jülich, Federal Republic of Germany.

0305-4470/92/113133+10$04.50

transformation, which maps in a special manner, the high-temperature behaviour of a model to the low-temperature one. Later on, this device was generalized, for example to gauge models [3, 4, 15]. Usually, there exists one *self-dual* temperature, defined as the fixed point of the transformation. If one knows (or assumes) that the model undergoes one, and only one, phase transition with respect to the tuned temperature then the transition must occur at the self-dual temperature. Some years later, a transformation for vertex models was constructed [12, 16], which shared some properties with duality transformations, and indeed covered some of the latter [16]. Recently [18], a renewed interest in this (generalized) *weak-graph transformation* has focused on isotropic vertex models, which contain the Ising model in an external field as a special case.

In this paper we will discuss the weak-graph transformation on regular lattices with even coordination number for vertex models which respect a completely different and far less restrictive symmetry. We obtain the fixed points of the transformation and investigate for some models whether there are phase transitions at the fixed points. The paper is organized as follows. Section 2 is devoted to a brief definition of the vertex model and introduction of notations. In section 3 we derive the weak-graph transformation and its main properties, and propose a calculation method for general coordination numbers of the underlying lattice. As an example, we continue the calculations to definite results for coordination number six in section 4. Sections 5 and 6 present applications of these results. In the former we discuss a gauge-like Ising model on a three-dimensional SC lattice. The latter deals with the ice-type, zero-field ferroelectric model and contains a rigorous proof for first-order phase transitions in this model. Section 7 is reserved for a discussion and outlook.

2. Definition of the model

This paper deals with vertex models on six-coordinated lattices ($q = 6$) with periodic boundary conditions. However, the method used [5] is, in principle, applicable to lattices of any *even* coordination number q. The dynamical variables on the lattice bonds are allowed to take two values: $\alpha = \pm 1$. The statistical weight $\omega(\alpha)$ of a vertex depends on q spins $\boldsymbol{\alpha} := (\alpha_1, \ldots, \alpha_q)$ on the bonds that make up the vertex. The only assumption made on the model throughout this paper is the symmetry of the vertex weights with respect to flipping all spins

$$\omega(-\alpha_1, \ldots, -\alpha_q) = \omega(\alpha_1, \ldots, \alpha_q). \tag{1}$$

This reduces the number of independent model parameters to 2^{q-1}.

3. The method used

The *generalized weak-graph transformation* [12, 16] for $q = 6$

$$\omega^*(\beta_1, \ldots, \beta_6) = \sum_{\alpha_1, \ldots, \alpha_6} V_{\beta_1\alpha_1} \cdots V_{\beta_3\alpha_3} \omega(\alpha_1, \ldots, \alpha_6)(V^{-1})_{\alpha_4\beta_4} \cdots (V^{-1})_{\alpha_6\beta_6}. \tag{2}$$

is a (multi-)linear mapping in parameter space which leaves the partition function

$$Z = \sum_{\{\alpha\}} \prod_{\text{vertices}} \omega(\alpha) \tag{3}$$

invariant. Here, V may be any regular (2×2) matrix. To make V easily invertible (namely by transposition), we take it to be an orthogonal matrix. The transformation then has, for general q, the symmetric form

$$\omega^*(\beta_1, \ldots, \beta_q) = \sum_{\alpha_1,\ldots,\alpha_q} V_{\beta_1\alpha_1} \ldots V_{\beta_q\alpha_q} \omega(\alpha_1, \ldots, \alpha_q). \tag{4}$$

As we want the transformation to be similar to duality transformations, which are involutions, we take $\det(V)$ to be -1 rather than $+1$ [16]. From now on, V will be parametrized by a (real) parameter y

$$V(y) = \frac{1}{\sqrt{1+y^2}} \begin{pmatrix} 1 & y \\ y & -1 \end{pmatrix}. \tag{5}$$

In this and the next section we obtain, for the model defined in the previous section, the manifold of those points which remain fixed under the weak-graph transformation. This manifold will turn out to be independent of the transformation-parameter y, i.e. the same for all V we consider (except for $V = 1$). The eigenvectors of $V(y)$ are easily obtained to be [5]

$$\psi(\epsilon) = \begin{pmatrix} y \\ z_\epsilon \end{pmatrix} \qquad z_\epsilon := \epsilon\sqrt{1+y^2} - 1 \tag{6}$$

corresponding to eigenvalues $\epsilon = +1$ and $\epsilon = -1$ respectively. The upper (lower) component of ψ will correspond to $\alpha = +1(-1)$ for convenience. The transformation (4) may be viewed [5, 6] as a q-fold tensor product $W := V \otimes \cdots \otimes V$ of V matrices. Every tensor product of eigenvectors of V belonging to eigenvalues $\epsilon_1, \ldots, \epsilon_q$

$$\Psi(\epsilon) := \psi(\epsilon_1) \otimes \ldots \otimes \psi(\epsilon_q) \tag{7}$$

is an eigenvector of W corresponding to the eigenvalue $\lambda(\epsilon) = \prod_{i=1}^q \epsilon_i = (-1)^{n_-(\epsilon)}$, where $n_-(\epsilon)$ denotes the number of negative components in ϵ, i.e. the number of $\epsilon_i = -1$. In conversion to the eigenvector property, the tensor products (7) form bases for the (two) eigenspaces of W. The space of vertex weight vectors ω invariant under the weak-graph transformation is thus spanned by all 2^{q-1} eigenvectors $\Psi(\epsilon)$ belonging to the eigenvalue $\lambda(\epsilon) = +1$. Because of the orthogonality property of the eigenvectors this space is equivalently characterized by the 2^{q-1} eigenvectors $\Psi(\epsilon)$ with eigenvalue $\lambda(\epsilon) = -1$. These (normal vectors) are $\Psi(\epsilon)$ with $\epsilon \in E_1 := \{\epsilon | \prod_{i=1}^q \epsilon_i = -1\}$, i.e. they are characterized by an *odd* number $n_-(\epsilon)$ of negative ϵ_i's. The normal vectors will be used to identify the space of fixed points for all parameter values $y \neq 0$ ($y = 0$ yields a trivial mapping).

The normal vectors make up a system of homogeneous linear equations for the vertex weights: $A\omega = 0$. Using (6) and (7) one obtains for the coefficients

$$A_{\epsilon,\alpha} := \Psi_\alpha(\epsilon) = y^{n_+(\alpha)} \prod_{i:\alpha_i=-1} z_{\epsilon_i} \qquad \epsilon \in E_1. \tag{8}$$

Two rows of the matrix $\mathbf{A}$ corresponding to row indices ϵ, ϵ' will be called a *complementary pair*, if $\epsilon_i = -\epsilon_i'$ for all $i = 1, \ldots, q$, i.e. if the row indices differ in *all* places from one another. The system of equations is now transformed by summing and subtracting respectively the two equations of each complementary pair. The index set for ϵ will be half of E_1. The coefficients read explicitly

$$A^{\pm}_{\epsilon,\alpha} := A_{\epsilon,\alpha} \bullet A_{-\epsilon,\alpha} = y^{n_+(\alpha)} \left(z_+{}^k z_-{}^l \pm z_-{}^k z_+{}^l \right). \tag{9}$$

Here $k := n_+(\epsilon|n_-(\alpha))$ and $l := n_-(\epsilon|n_-(\alpha)) = n_-(\alpha) - k$ denote the number of positive and negative ϵ_i's respectively amongst those from places $i = 1, \ldots, q$ which have negative α_i. Any power of z_ϵ can be linearized

$$z_\epsilon{}^k = P_k(y^2) + Q_k(y^2) z_\epsilon \qquad k \in \{0, 1, 2, \ldots\} \tag{10}$$

with polynomials P_k, Q_k of $\lfloor k/2 \rfloor$th order as coefficients. This can be shown by induction using the defintion (6) of z_ϵ. The polynomials are defined recursively: $P_0 = 1, Q_0 = 0, P_k = y^2 Q_{k-1}, Q_k = P_{k-1} - 2Q_{k-1}$. Using $z_+ + z_- = -2$ and $z_+ z_- = -y^2$ one simplifies the expression in (9)

$$z_+{}^k z_-{}^l \pm z_-{}^k z_+{}^l \begin{cases} 2[P_k(Q_{l+1} + Q_l) - Q_k(P_{l+1} + P_l)] \\ (z_- - z_+)[P_k Q_l - Q_k P_l]. \end{cases} \tag{11}$$

Because the system of linear equations under consideration is a homogeneous one, the α-independent factors 2 and $z_- - z_+ \neq 0$ may be cancelled and the second set of equations may be subtracted from the first:

$$A^{\pm}_{\epsilon,\alpha} = y^{n_+(\alpha)} \begin{cases} P_k Q_{l+1} - Q_k P_{l+1} \\ P_k Q_l - Q_k P_l. \end{cases} \tag{12}$$

Note, that the coefficients for the fixed-point conditions have become just polynomials in y. Any occurrence of z_ϵ has vanished.

4. The manifold of fixed points in case of $q = 6$

Now turn to the case $q = 6$, i.e. the general $(2^q = 64)$-vertex model and consider $A^{\pm}\omega = 0$. Using spin-flip symmetry (1), we performed linear transformations with the rows of the matrices $\mathbf{A}^+$ and $\mathbf{A}^-$ and were able to simplify them in such a way, that the fixed-point equations become manageable. The results are in the following, whereas details of the calculations will be published elsewhere.

The vertex weights $\omega(\alpha)$ will be denoted by a, b, c, d if they contain $n_-(\alpha) = 0, 1, 2, 3$ negative arguments α_i, respectively. Their position will augment the character by one or more indices:

$$\begin{aligned} a &= \omega(+ + + + + +) \\ b_r &= \omega(+ \ldots + \overset{r}{-} + \ldots +) \\ c_{rs} \equiv c_{sr} &= \omega(+ \ldots + \overset{r}{-} + \ldots + \overset{s}{-} + \ldots +) \qquad r < s \\ d_{rst} &= \omega(+ \ldots + \overset{r}{-} + \ldots + \overset{s}{-} + \ldots + \overset{t}{-} + \ldots +) \qquad r < s < t \end{aligned} \tag{13}$$

where $r, s, t \in \{1, \ldots, 6\}$. The remaining vertex weights (with $n_-(\alpha) > 3$) are determined by spin-flip symmetry (1), which, in addition, leaves vertex weights d corresponding to half of the indicated indices independent, only: $d_{rst} = d_{uvw}$ if $\{r, s, t, u, v, w\} = \{1, \ldots, 6\}$. Using this terminology, the transformed equations A^+ reduce to

$$(y^2 - \tfrac{1}{3})b_1 = 0 \qquad b_1 = b_i = -d_{rst} \qquad i, r, s, t \in \{1, \ldots, 6\} \qquad r < s < t \tag{14}$$

so that, for non-negative vertex weights, all non-vanishing weights have to satisfy the constraint

$$\prod_{i=1}^{6} \alpha_i = 1. \tag{15}$$

The remaining 16 independent weights of the spin-flip symmetric (now 32-)vertex model on the manifold of fixed points are restricted by the transformed equations $\mathbf{A}^-$

$$a = \sum_{s:s \neq r} c_{rs} \qquad \text{for all } r \in \{1, \ldots, 6\} \tag{16}$$

$$\sum_{r,s \in F: r<s} c_{rs} = \sum_{r,s \in F': r<s} c_{rs} \qquad \text{for all } F \subset \{1, \ldots, 6\} \text{ with } |F| = 3 \tag{17}$$

where $F' = \{1, \ldots, 6\} \backslash F$ is the set complementary to F. By adding appropriate equations from (16), one can show that the equations in (17) are linearly dependent on (16) and, thus, can be cancelled. The six equations (16) are linearly independent. Thus the manifold of fixed points is ten-dimensional, described by (16). It is independent of y, i.e. the same for all matrices V we considered.

5. An equivalent Ising model

For the general ($q = 4$) 16-vertex model (VM) one finds a manifold of fixed points which is easily seen to be identical [5] with the critical manifold of Baxter's 8VM [1]. The subject of the following sections will be the coincidence of the corresponding manifolds in the spin-flip symmetric ($q = 6$) 32VM respecting the constraint (15), which was introduced in the previous sections. Now, when thinking of the 32VM as a generalization of the two-dimensional 8VM, it is natural to ask whether the former can be formulated in terms of spins on a three-dimensional lattice, quite by analogy with the situation for the latter in two dimensions [7, 17]. Indeed, we find by generalizing the methods of Kadanoff and Wegner [7], the following correspondence

$$Z_{\mathrm{I}}(K_{\bar{i}\bar{j}}, K_{\bar{i}j}, K_{i\bar{j}}, K_{ij}; L_i) = \text{combinatorial factor } \times Z_{\mathrm{VM}}(a; c_{rs}) \tag{18}$$

where Z_{I} denotes the partition function of an Ising-type model with 15 interaction strengths $K_{\bar{i}\bar{j}}, K_{\bar{i}j}$ $K_{i\bar{j}}, K_{ij} (i < j = 1, \ldots, 3)$ and $L_i (i = 1, \ldots, 3)$, which are

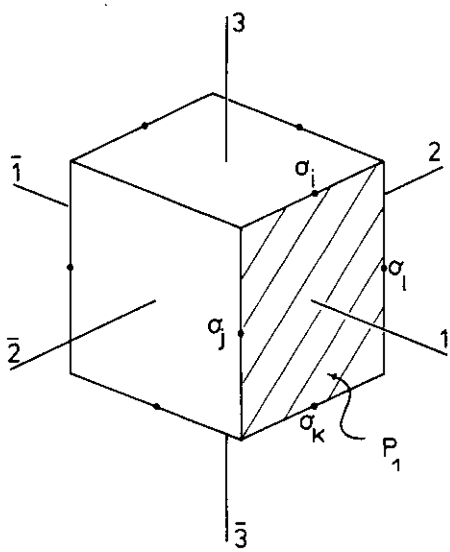

Figure 1. The four σ-spins on edges around the plaquette P_1, yielding the product-spin $\alpha_1 = \sigma_i \sigma_j \sigma_k \sigma_l$.

unique functions of the 16-vertex weights $a, c_{rs}(r < s = 1, \ldots, 6)$ of the corresponding vertex model (see the following).

To see this, place spins $\sigma = \pm 1$ at the edges of a three-dimensional SC lattice, as in figure 1, and assume periodic boundary conditions. Now, instead of dealing with these σ-spins, we construct product spins, which are living on plaquettes (faces) of the lattice. Denoting the plaquettes of a cube by P_i as is indicated in figure 1, we introduce the product spins as

$$\alpha_i = \prod_{l \in P_i} \sigma_l \qquad i = \bar{1}, \ldots, \bar{3}, 1, \ldots, 3 \tag{19}$$

where an index i ($\bar{i}$) refers to the plaquette orthogonal to the lattice direction i on the front (back) of the cube, as seen in a fixed orientation of the lattice direction. The new α-variables are again spins with values ± 1. Moreover, we notice that the product of all plaquette spins situated on the six plaquettes of a cube satisfy the relation

$$\prod_{i=1}^{3} \alpha_{\bar{i}} \alpha_i = 1. \tag{20}$$

The most general Hamiltonian (up to a zero of energy), which can be written as a sum over cubes and is α-spin-flip symmetric, contains interactions between nearest and next-nearest neighbour spins of the plaquette-type

$$-\beta E_I = \sum_{\text{cubes}} \left\{ \sum_{i<j=1}^{3} \left(K_{\bar{i}\bar{j}} \alpha_{\bar{i}} \alpha_{\bar{j}} + K_{\bar{i}j} \alpha_{\bar{i}} \alpha_j + K_{i\bar{j}} \alpha_i \alpha_{\bar{j}} + K_{ij} \alpha_i \alpha_j \right) \right.$$
$$\left. + \sum_{i=1}^{3} L_i \alpha_{\bar{i}} \alpha_i \right\}. \tag{21}$$

This model contains six-spin interactions between the original σ-spins round every two adjacent plaquettes of a cube (interaction strengths K) and, moreover, eight-spin interactions between σ-spins on two opposite plaquettes (interaction strengths L). The partition function is given by

$$\mathcal{Z}_I = \sum_{\{\sigma\}} \exp(-\beta E_I(\{\sigma\})). \tag{22}$$

We then argue just as in the two-dimensional case studied in [7] and conclude that (18) holds, provided the vertex weights respecting (20) are taken as

$$\omega(\alpha_{\bar{1}}, \ldots, \alpha_3) = \exp\left\{ \sum_{i<j=1}^{3} \left(K_{\bar{i}\bar{j}}\alpha_{\bar{i}}\alpha_{\bar{j}} + K_{\bar{i}j}\alpha_{\bar{i}}\alpha_j + K_{i\bar{j}}\alpha_i\alpha_{\bar{j}} + K_{ij}\alpha_i\alpha_j \right) \right. \\ \left. + \sum_{i=1}^{3} L_i \alpha_{\bar{i}}\alpha_i \right\}. \tag{23}$$

The 32VM with spin-flip symmetry has, thus, been shown to be equivalent to the Ising-type model (19) and (21) of spins σ on the edges of a three-dimensional SC lattice.

It is interesting to note that in the case of isotropic couplings, the vertex weights are related to the two remaining coupling constants K and L by

$$a = \exp(3L + 12K) \\ c_{rs} = \begin{cases} \exp(3L - 4K) & \text{for } (r,s) = (\bar{1},1), (\bar{2},2), (\bar{3},3) \\ \exp(-L) & \text{else.} \end{cases} \tag{24}$$

The six equations (16), which define a manifold of fixed points, now reduce to a single one: $\sinh(8K) = 2\exp(-4L - 4K)$. We conjecture this line in the (K, L)-plane to represent the exact loci of phase transitions for the isotropic model. Monte Carlo simulations performed by us support this conjecture. The internal energy and the magnetization both seem to have a discontinuity at the expected couplings. Thus the Ising model studied above undergoes a first-order phase transition at these points.

6. Proof of a first-order phase transition in the ice rule sector

In the last sections we investigated a 32VM with special spin-flip symmetric vertex-weights $\omega(\alpha_{\bar{1}}, \ldots, \alpha_3)$, where the variables $\alpha = \pm 1$ satisfy the constraint (20). Now we choose another point of view and interpret the variables as arrow-spins associated with the six edges round a vertex, as in figure 2. We have generalized the usual convention practised in two-dimensional ice-type models that an arrow pointing in one of the three lattice directions $i = 1, \ldots, 3$ is related to a variable $+1$ (-1 otherwise).

Up to now we have dealt with vertex models satisfying the more general constraint, which allows all configurations with an odd number of arrows into and out of each site of the lattice, in mathematical terms formulated in (20). In this section we restrict ourselves to a *KDP-type ice rule*, where at each vertex of the lattice, there

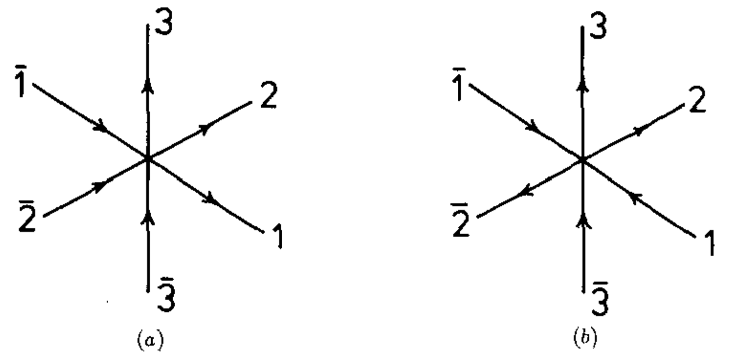

Figure 2. Two examples of arrow configurations $(\alpha_{\bar{1}}, \alpha_{\bar{2}}, \alpha_{\bar{3}}, \alpha_1, \alpha_2, \alpha_3)$ introduced in the text, corresponding to weights (a) $a = \omega(+++++++)$ and (b) $c_{\bar{2}1} = \omega(+-+-++)$. They both respect the generalized ice rule as defined in the text.

are three arrows in and three arrows out. Note that this rule is adapted to hydrogen-bonded cubic crystals and ensures local electrical neutrality. The arrow configurations colliding with the ice rule are those with weights c_{ij} and $c_{\bar{i}\bar{j}}$, which we have now to set to zero. If we interpret each arrow as carrying an electric dipole moment, we can divide the remaining non-zero weights into two classes. In one class we assemble all configurations with a net dipole pointing along a diagonal direction, i.e. the vertices a and $c_{\bar{i}i}$, associated with dipoles pointing parallel to the directions d and $d - 2e_i$, respectively. Here $d = (1,1,1)$ and e_i is the ith unit vector. In the second class we group together all configurations which have a net dipole parallel to the axial directions e_k, i.e. the vertices $c_{\bar{i}j}$ with $\{i,j,k\} = \{1,2,3\}$. Because axial dipoles are associated with two weights each, we demand $c_{\bar{i}j} = c_{\bar{j}i}$ for $i \neq j$.

Now we proceed just as in the theory of the various two-dimensional ferroelectric models studied in the literature [11]. We postulate an isotropic c-axis which coincides with any one of the eight allowed dipole axes (axial or diagonal). If the configuration with a net dipole pointing along the crystal axis is given the lowest energy, then at low temperatures there is a tendency for a spontaneous polarization, thus resulting in a (zero-field) ferroelectric model. In what follows, we choose a c-axis parallel to d and, thus, favour the weight a. The most general energy assignment then is

$$\begin{aligned} a &= 1 \\ c_{\bar{j}j} &= \exp(-\beta\epsilon_j) = \exp(-K_j) \qquad j = 1, \ldots, 3 \\ c_{\bar{i}j} &= c_{\bar{j}i} = \exp(-\beta\mu_k) = \exp(-L_k) \qquad \{i,j,k\} = \{1,2,3\} \end{aligned} \tag{25}$$

where $\epsilon_j, \mu_j > 0 (j = 1,2,3)$ are anisotropy parameters of the model [9, 13, 14]. The parameters K and L must not be confused with the couplings of the Ising model introduced in the previous section.

Our aim in this section is to prove that the critical manifold of the KDP model defined in (25) coincides with the manifold of fixed points, which is now given by (cf (16))

$$1 = \exp(-K_j) + \sum_{i:i \neq j} \exp(-L_i) \qquad j = 1, \ldots, 3. \tag{26}$$

On this manifold we use the high-temperature expansion [10]

$$Z_N = \sum_{\text{states}} \prod_{(mn)} \frac{1 + \gamma_{mn}(\xi_m)\gamma_{nm}(\xi_n)}{2} \prod_n \omega(\xi_n) \tag{27}$$

where the sum is over all arrow configurations, the second product is over all vertices n of the lattice, and the first product is over all edges (mn). The variable ξ_n specifies the vertex configuration at vertex n: $\omega(\xi_n) = \exp(-K_j), \exp(-L_k), 1$ or 0, depending on whether the vertex configuration ξ_n has been assigned an energy $\epsilon_j, \mu_k, 0$ or does not respect the ice rule. Finally, γ_{mn} is a characteristic function taking on the value $+1$ (-1) if the arrow on the edge (mn) has a positive (negative) projection on to the c-axis. We expand the first product in (27) and think of each term as representing a graph of lines on the lattice, where every factor $\gamma_{mn}\gamma_{nm}$ corresponds to a line between vertices m and n. The partition function can be written as

$$\mathcal{Z}_N = 2^{-Nq/2} \sum_{\text{graphs}} \prod_n b(\Gamma_n) \tag{28}$$

where Γ_n denotes the configuration of lines radiating from vertex n, and

$$b(\Gamma_n) = \sum_{\xi_n} \omega(\xi_n) \prod_{m\in\Gamma_n} \gamma_{nm}(\xi_m). \tag{29}$$

The crucial point then is that all weights b with an excess of ± 2 lines in the c-direction vanish on the manifold of fixed points (26). This enables the free energy and the entropy per site to be calculated in the thermodynamic limit $N \to \infty$ on the manifold (26), on which we define the temperature T_c using (25) with fixed energies ϵ_j, μ_k,

$$\left.\frac{F}{N}\right|_{T_c} \equiv -k_B T_c \lim_{N\to\infty} \left(\frac{1}{N} \ln \mathcal{Z}|_{T_c} \right) = 0 \tag{30}$$

$$\left.\frac{S}{N}\right|_{T_c} = \frac{k_B}{4} \sum_{j=1}^{3} (K_j e^{-K_j} + 2L_j e^{-L_j}) > 0. \tag{31}$$

Now it is an easy step to the final conclusion that the manifold (26) is a critical manifold. To this end we note that the ground state is doubly degenerate and its energy equal to zero by (25), thus $F/N = S/N = 0$ at $T = 0$ in the thermodynamic limit. Combining with (30) and the monotony property $\partial F/\partial T \equiv -S \leqslant 0$, the free energy and the entropy must vanish on the whole interval $0 \leqslant T \leqslant T_c$. This establishes the frozen order in the system below T_c. S/N becomes discontinuous at T_c and shows a finite jump according to (31), if at least one of the energies ϵ_j or μ_k is positive. This confirms that a first-order phase transition occurs on (26), and that, at least in the ice rule sector, the fixed points of the 32VM are phase transition points.

7. Conclusion and outlook

For general spin-flip symmetric (two-state) vertex models on lattices of even coordination number we have investigated the generalized weak-graph transformation. The manifold of fixed points, obtained in the case of a coordination number 6, is characterized by two types of conditions. The first one demands all vertex weights with an odd number of arguments $+1$ (-1) to vanish, thus yielding a 32-vertex model. The second one consists of six equations, linear in the 16-vertex weights remaining. As an application, we discussed two models contained in the 32-vertex model. The first one is a model of Ising spins on the edges of the three-dimensional SC lattice, coupled by gauge-invariant six- and eight-spin interactions. In the case of isotropic couplings, we find a line of fixed points, which, according to Monte-Carlo simulations [5], is a line of phase transitions. The second model discussed is the ice-type, zero-field ferroelectric model on six-coordinated lattices. Again, the fixed points specialized to this case coincide with points of first-order phase transitions.

One of the surprising properties of Baxter's eight-vertex model is the occurrence of parameter-dependent critical exponents [1]. Since the spin-flip symmetric vertex model investigated in this paper is a natural generalization of the former model to q-coordinated lattices, one is led to ask the question whether the loss of universality will also occur in higher dimensions. Unfortunately, the methods used in this paper are not able to yield critical indices, so that the question can only be answered with the help of numerical methods or with more elaborate theoretical means. There is, however, a hint against the breakdown of universality in lattices with $q \geqslant 6$: the three-dimensional Ashkin–Teller model on a SC lattice has been investigated by Ditzian *et al* [2] and is found to be a model with universal properties, quite in contrast to the Ashkin–Teller model on the square lattice which can be interpreted as a staggered eight-vertex model. This fact suggests that the third dimension restores universality which is broken in two dimensions.

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